MATH 181 - Final Exam
December 15, 2016

Name:______________________________________

1. Compute the following limits:

a. $\lim\limits_{x\to-\infty} \dfrac{6x^5 - 4x^3 + 3x^2 - x + 1}{(1 - x^5)}$

b. $\lim\limits_{x\to 2} \dfrac{x^2 + x - 6}{6 - 5x + x^2}$

c. $\lim\limits_{x\to 0} \dfrac{1 - \cos(4x)}{x^2}$

d. $\lim\limits_{x\to 0} \dfrac{\sqrt{1 + 2x} - \sqrt{1 - 4x}}{x}$

e. $\lim\limits_{x\to\infty} x^{\frac{1}{x}}$

2. Find the derivatives of the following functions:

a. $f(x) = 1 + x^{-\frac{2}{3}} - \frac{1}{x} + e^{\pi x} + \cos(3x) + \pi^2$

b. $g(x) = \dfrac{Ln(2x+3)}{\sin(5x)}$

c. $h(t) = \tan^{-1}(2t)\sqrt{2t-1}$

d. $f(x) = \lim\limits_{h \to 0} \dfrac{2^{x+h} - 2^x}{h}$

e. $f(x) = \displaystyle\int_1^{x^2} \sec(t)\ dt$

3. Compute the following integrals:

a. $\displaystyle\int 1 + x^{-\frac{2}{3}} - \frac{1}{x} + e^{\pi x} + \cos(3x) + \pi^2$

b. $\displaystyle\int x\sqrt{x^2+1}\, dx$

c. $\displaystyle\int x\sqrt{x+1}\, dx$

d. $\displaystyle\int_0^1 \frac{e^x}{1+e^x}\, dx$

e. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x\cos(x) - \sin x}{x^2}\, dx$

4. Sketch the curve $f(x) = 5x^{\frac{2}{3}} - 2x^{\frac{5}{3}}$ by completing the following:

a. Find all/any intercepts(s)

b. Find the intervals of increasing, decreasing, and all/any local extreme points

c. Determine the concavity of f and find all/any points of inflection

d. Sketch the graph of $y = f(x)$

5. Find the equation of the line that is tangent to the curve $x^2 + 2xy + 4y^2 = 12$ at the point $(2, 1)$

6. A ladder $10ft$ long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of $1\frac{ft}{sec}$, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is $6ft$ from the wall?

7. If 1200 cm^2 of material is available to make a box with a square base and open top, find the largest possible volume of the box.

1a) $\lim_{x\to -\infty} \frac{6x^5-4x^3+3x^2-x+1}{(1-x^5)} \cdot \frac{\frac{1}{x^5}}{\frac{1}{x^5}}$

$$= \lim_{x\to-\infty} \frac{\frac{6x^5}{x^5} - \frac{4x^3}{x^5} + \frac{3x^2}{x^5} - \frac{x}{x^5} + \frac{1}{x^5}}{\frac{1}{x^5} - \frac{x^5}{x^5}}$$

$$= \lim_{x\to-\infty} \frac{6 - \frac{4}{x^2} + \frac{3}{x^3} - \frac{1}{x^4} + \frac{1}{x^5}}{\frac{1}{x^5} - 1}$$

$$= \frac{6 - \cancel{\frac{4}{\infty^2}}^{\to 0} + \cancel{\frac{3}{\infty^3}}^{\to 0} - \cancel{\frac{1}{\infty^4}}^{\to 0} + \cancel{\frac{1}{\infty^5}}^{\to 0}}{\cancel{\frac{1}{\infty^5}}^{\to 0} - 1}$$

$$= \frac{6-0+0-0+0}{0-1} = \frac{6}{-1} = \boxed{-6}$$

1b) $\lim_{x\to 2} \frac{x^2+x-6}{6-5x+x^2}$

$$= \lim_{x\to 2} \frac{(x+3)\cancel{(x-2)}}{(x-3)\cancel{(x-2)}}$$

$$= \lim_{x\to 2} \frac{(x+3)}{(x-3)}$$

$$= \frac{2+3}{2-3} = \frac{5}{-1} = \boxed{-5}$$

1c) $\lim_{x\to 0} \frac{1-\cos(4x)}{x^2}$ "$\frac{0}{0}$"

$$\overset{LH}{=} \lim_{x\to 0} \frac{4\sin(4x)}{2x} \quad \text{"}\frac{0}{0}\text{"}$$

$$\overset{LH}{=} \lim_{x\to 0} \frac{16\cos(4x)}{2} = \frac{16\cos(4\cdot 0)}{2} = \frac{16(1)}{2} = \boxed{8}$$

1d) $\lim_{x\to 0} \frac{\sqrt{1+2x}-\sqrt{1-4x}}{x}$ "$\frac{0}{0}$"

$$\overset{LH}{=} \lim_{x\to 0} \frac{\cancel{\tfrac{1}{2}}(1+2x)^{-1/2}\cdot \cancel{2} - \cancel{\tfrac{1}{2}}(1-4x)^{-1/2}\cdot \cancel{(-4)}^{-2}}{1}$$

$$= \lim_{x\to 0} (1+2x)^{-1/2} - (-2)(1-4x)^{-1/2}$$

$$= (1+0)^{-1/2} + 2(1-0)^{-1/2} = 1+2(1) = 1+2 = \boxed{3}$$

1e) $\lim_{x\to\infty} x^{\frac{1}{x}}$

let $y = x^{\frac{1}{x}}$

$\ln|y| = \frac{1}{x}\ln|x|$

$= \lim_{x\to\infty} \ln y$

$= \lim_{x\to\infty} \frac{\ln x}{x}$ "$\frac{\infty}{\infty}$"

$\overset{LH}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$

$= \lim_{x\to\infty} \frac{1}{x} = 0$

so $\lim_{x\to\infty} y = e^0 = \boxed{1}$

2a) $f(x) = 1 + x^{-2/3} - \frac{1}{x} + e^{\pi x} + \cos(3x) + \pi^2$

$f'(x) = 0 - \frac{2}{3}x^{-\frac{5}{3}} + x^{-2} + \pi e^{\pi x} - 3\sin(3x) + 0$

$\boxed{f'(x) = \frac{-2}{3x^{5/3}} + \frac{1}{x^2} + \pi e^{\pi x} - 3\sin(3x)}$

2b) $g(x) = \frac{\ln(2x+3)}{\sin(5x)}$

$g'(x) = \frac{\frac{1}{2x+3}\cdot 2 \cdot \sin(5x) - \ln(2x+3)\cdot 5\cos(5x)}{[\sin(5x)]^2}$

$= \frac{\frac{2\sin(5x)}{2x+3} - 5\ln(2x+3)\cos(5x)}{\sin^2(5x)}$

$= \frac{\frac{2\sin(5x)}{2x+3}}{\sin^2(5x)} - \frac{5\ln(2x+3)\cos(5x)}{\sin^2(5x)}$

$= \frac{2\cancel{\sin(5x)}}{(2x+3)\sin^{\cancel{2}}(5x)} - \frac{5\ln(2x+3)\cos(5x)}{\sin^2(5x)}$

$= \boxed{\frac{2}{(2x+3)\sin(5x)} - \frac{5\ln(2x+3)\cos(5x)}{\sin^2(5x)}}$

2c $h(t)=\tan^{-1}(2t)\cdot\sqrt{2t-1}$

$$h'(t)=\frac{1}{(2t)^2+1}\cdot 2\cdot\sqrt{2t-1}+\cancel{\tfrac{1}{2}}(2t-1)^{-1/2}\cdot\cancel{2}\cdot\tan^{-1}(2t)$$

$$=\boxed{\frac{2\sqrt{2t-1}}{4t^2+1}+\frac{\tan^{-1}(2t)}{\sqrt{2t-1}}}$$

2d) $f(x)=\lim_{h\to 0}\left(\frac{2^{x+h}-2^x}{h}\right)$

$$=\lim_{h\to 0}\frac{2^x(2^h-1)}{h}$$

$$=2^x\lim_{h\to 0}\frac{2^h-1}{h}\quad \text{"}\frac{0}{0}\text{"}$$

$$\overset{LH}{=}2^x\lim_{h\to 0}\frac{2^h\ln(2)}{1}=2^x\left(\cancel{2^0}^{1}\ln(2)\right)$$

$$f(x)=2^x\ln|2|$$

$$f'(x)=2^x\cdot\ln|2|\cdot\ln|2|$$

$$=\boxed{2x(\ln|2|)^2}$$

2e) $f(x)=\int_1^{x^2}\sec(t)\,dt$

$$f'(x)=\sec(x^2)\cdot\frac{d}{dx}[x^2]\quad \text{by FTC 1}$$

$$\boxed{f'(x)=\sec(x^2)\cdot 2x}$$

3a) $\int 1+x^{-2/3}-\frac{1}{x}+e^{\pi x}+\cos(3x)+\pi^2$

$$=\boxed{x+3x^{1/3}-\ln|x|+\frac{1}{\pi}e^{\pi x}+\frac{1}{3}\sin(3x)+\pi^2x+C}$$

3b) $\int x\sqrt{x^2+1}\,dx$

$= \int x(x^2+1)^{1/2}\,dx$

$= \frac{1}{2}\int 2x(x^2+1)^{1/2}\,dx$ let $u = x^2+1$

$= \frac{1}{2}\int u^{1/2}\,du$ $du = 2x\,dx$

$= \frac{1}{2}\left(\frac{\cancel{2}}{3}u^{3/2}\right)$

$= \frac{1}{3}(x^2+1)^{3/2} + C$

$= \boxed{\frac{\sqrt{(x^2+1)^3}}{3} + C}$

3c) $\int x\sqrt{x+1}\,dx$

let $u = x+1$

$u-1 = x$

$du = dx$

$= \int x(x+1)^{1/2}\,dx$

$= \int (u-1)(u-1+1)^{1/2}\,du$

$= \int (u-1)(u^{1/2})\,du$

$= \int u^{3/2} - u^{1/2}\,du$

$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$

$= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C = \boxed{\frac{2\sqrt{(x+1)^5}}{5} - \frac{2\sqrt{(x+1)^3}}{3} + C}$

3d) $\int_0^1 \frac{e^x}{1+e^x}\,dx$ let $u = 1+e^x$

$du = e^x\,dx$

$= \int_{x=0}^{x=1} \frac{1}{u}\,du$

$= \ln|u|\Big]_{x=0}^{x=1}$

$= \ln|1+e^x|\Big]_0^1$

$= \ln|1+e^1| - \ln|1+e^0|$

$= \ln|1+e) - \ln|1+1|$

$= \ln|1+e| - \ln|2) = \boxed{\ln\left(\frac{1+e}{2}\right)}$

3e) $\int_{\pi/4}^{\pi/2} \frac{x\cos(x)-\sin(x)}{x^2}\,dx$

$\frac{x\cos(x)-\sin(x)}{x^2}$ is in the form of $\frac{f'(x)g(x)-g'(x)f(x)}{g^2(x)}$

when $g(x)=x$ & $f(x)=\sin(x)$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{x\cos(x)-\sin(x)}{x^2}$$

therefore, $\int_{\pi/4}^{\pi/2} \frac{x\cos(x)-\sin(x)}{x^2} = \frac{f(x)}{g(x)} = \left.\frac{\sin(x)}{x}\right|_{\pi/4}^{\pi/2}$

$$= \frac{\sin(\pi/2)}{\pi/2} - \frac{\sin(\pi/4)}{\pi/4}$$

$$= \frac{2}{\pi}\cdot 1 - \frac{\sqrt{2}}{\cancel{2}}\cdot\frac{\cancel{4}\,2}{\pi}$$

$$= \frac{2}{\pi} - \frac{2\sqrt{2}}{\pi}$$

$$= \boxed{\frac{2-2\sqrt{2}}{\pi}}$$

4a) $f(x)=5x^{2/3}-2x^{5/3}$

$f(x)=x^{2/3}(5-2x)$

x-intercepts $0=x^{2/3}(5-2x)$

$0=x^{2/3}$ $\quad x=0$

$0=5-2x$

$2x=5$

$x=\frac{5}{2}$

$\boxed{\text{x-intercepts @: } x=0,\ x=\frac{5}{2}}$

y-intercepts $y=x^{2/3}(5-2x)$

$y=0^{2/3}(5-2(0))$

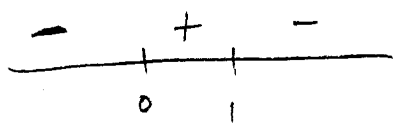
$y=0$

$\boxed{\text{y-intercept @: } y=0}$

4 b) $f'(x) = \frac{10}{3}x^{-\frac{1}{3}} - \frac{10}{3}x^{\frac{2}{3}}$

$0 = \frac{10}{3}x^{-1/3}(1-x)$

$x = 0, 1$



Dec $(-\infty, 0) \cup (1, \infty)$

INC $(0,1)$

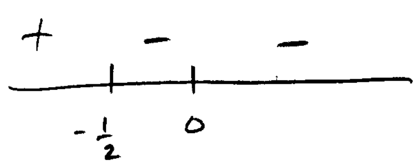
LOCAL EXT: $(0,0)$, $(1,3)$

4 c) $f''(x) = -\frac{10}{9}x^{-\frac{4}{3}} - \frac{20}{9}x^{-\frac{1}{3}}$

$0 = -\frac{10}{9}x^{-4/3} - \frac{20}{9}x^{-1/3}$

$-\frac{10}{9}x^{-\frac{4}{3}}(1+2x)$

$x = 0, -\frac{1}{2}$

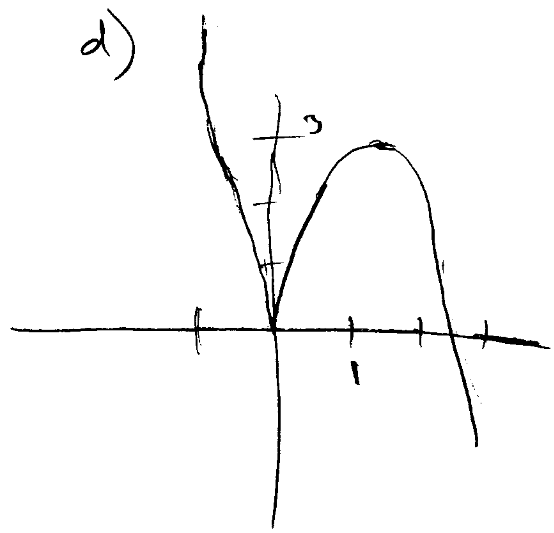


Concave up: $(-\infty, -\frac{1}{2})$

Concave down $(-\frac{1}{2}, \infty)$

POI @ $(0,0)$

d)



5) $\frac{d}{dx}(x^2+2xy+4y^2)=\frac{d}{dx}(12)$

$2x+2y+2xy'+8yy'=0$

$\frac{2}{2}(x+y+xy'+4yy')=\frac{0}{2}$

$xy'+4yy'=-x-y$

$\frac{y'(x+4y)}{(x+4y)}=\frac{-x-y}{(x+4y)}$

$y'=\frac{-x-y}{x+4y}$

$m_{tan}=\frac{-2-1}{2+4(1)}=-\frac{1}{2}$

$y-1=\frac{-1}{2}(x-2)$

$\boxed{y=\frac{-1}{2}x+2}$

6) $1\frac{ft}{sec}=\frac{dx}{dt}$, Find $\frac{dy}{dt}$

$6^2+y^2=10^2$

$y^2=100-36$

$y=8$

$x^2+y^2=10^2$

$dt = 2x\left(\frac{dx}{dt}\right)+2y\left(\frac{dx}{dt}\right)=0$

$y\left(\frac{dy}{dt}\right)=-x\left(\frac{dx}{dt}\right)$

$\frac{dy}{dt}=\frac{-x}{y}\left(\frac{dx}{dt}\right) \rightarrow \frac{dy}{dt}=-\frac{6}{8}(1) \rightarrow \boxed{\frac{dy}{dt}=-\frac{3}{4}\ ft/sec.}$

7) $1200=L^2+4LH$

$\frac{1200-L^2}{4L}=\frac{4LH}{4L}$

$H=\frac{300}{L}-\frac{1}{4}L$

$\boxed{L=20,\ H=10}$

$V=L^2\left(\frac{300}{L}-\frac{L}{4}\right)$

$V=300L-\frac{L^3}{4}$

$V'=300-\frac{3L^2}{4}$

$V'=0=300-\frac{3L^2}{4}$

$\overset{100}{\cancel{300}}\cdot\frac{4}{\cancel{3}}=L^2$

$L^2=400$

$L=20$

$H=\frac{300}{20}-\frac{1}{4}\cdot 20=15-5=10$