Convergence Tests for Series

Test for Divergence $$\sum_{n=1}^{\infty} a_n$$	▪ If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges ▪ If $\lim_{n\to\infty} a_n = 0$, then inconclusive
Geometric Series $$\sum_{n=0}^{\infty} ar^{n-1}$$	▪ If $\lvert r\rvert < 1$, the series converges to $\frac{a}{1-r}$ ▪ If $\lvert r\rvert \geq 1$, then the series diverges
Integral Test $$\sum_{n=c}^{\infty} a_n \text{ where } c \geq 0 \text{ and } a_n = f(n) \text{ for all } n$$	▪ $f(n)$ must be continuous, positive, and decreasing ▪ If $\int_c^{\infty} f(x)dx$ converges, then the series converges ▪ If $\int_c^{\infty} f(x)dx$ diverges, then the series diverges
p-series $$\sum_{n=1}^{\infty} \frac{1}{n^p}$$	▪ If $p > 1$, then the series converges ▪ If $p \leq 1$, then the series diverges
Comparison Test $$\sum a_n \text{ and } \sum b_n \text{ where } 0 \leq a_n \leq b_n \text{ for all } n$$	▪ If $\sum b_n$ converges, then $\sum a_n$ converges ▪ If $\sum a_n$ diverges, then $\sum b_n$ diverges
Limit Comparison Test $$\sum a_n \text{ and } \sum b_n \text{ where } a_n, b_n > 0 \text{ and } \lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$$	▪ If $\sum b_n$ converges, then $\sum a_n$ converges ▪ If $\sum a_n$ diverges, then $\sum b_n$ diverges ▪ To find b_n consider only the terms of a_n that have the greatest effect on the magnitude
Alternating Series Test $$\sum_{n=1}^{\infty} (-1)^{n-1} b_n \text{ where } b_n > 0$$	▪ Converges if $0 < b_{n+1} < b_n$ for all n and $\lim_{n\to\infty} b_n = 0$
Absolute Value Test $$\sum a_n$$	▪ If $\sum\lvert a_n\rvert$ converges, then $\sum a_n$ converges ▪ If the series of absolute values $\sum\lvert a_n\rvert$ is convergent, then the series is ***absolutely convergent*** ▪ If the series is convergent but not absolutely convergent, then the series is ***conditionally convergent***
Ratio Test $$\sum a_n \text{ with } \lim_{n\to\infty} \frac{\lvert a_{n+1}\rvert}{\lvert a_n\rvert} = L$$	▪ If $L < 1$, then the series converges absolutely ▪ If $L > 1$ or L is infinite, then the series diverges ▪ If $L = 1$, then the test is inconclusive
Root Test $$\sum a_n \text{ with } \lim_{n\to\infty} \sqrt[n]{a_n} = L$$	▪ If $L < 1$, then the series converges absolutely ▪ If $L > 1$ or L is infinite, then the series diverges ▪ If $L = 1$, then the test is inconclusive

Flowchart for Convergence Tests for Series

Test for Divergence

Does $\lim_{n\to\infty} a_n = 0$?

- No → $\sum a_n$ Diverges
- Yes ↓

P-Series

Does $a_n = 1/n^p$, $n \geq 1$?

- Yes → Is $p > 1$
  - Yes → $\sum a_n$ Converges
  - No → $\sum a_n$ Diverges
- No ↓

Geometric Series

Does $a_n = ar^{n-1}$, $n \geq 1$

- Yes → Is $|r| < 1$?
  - Yes → $\sum_{n=1}^{\infty} a_n = \frac{a}{1-r}$
  - No → $\sum a_n$ Diverges
- No ↓

Alternating Series Test

Does $a_n = (-1)^n b_n$ or $a_n = (-1)^{n-1} b_n$, $b_n \geq 0$?

- Yes → Is $b_{n+1} \leq b_n$ & $\lim_{n\to\infty} b_n = 0$
  - Yes → $\sum a_n$ Converges
- No ↓

<u>Try one of the following Tests:</u>

Comparison Test

Pick $\{b_n\}$. Does $\sum b_n$ converge?

- Yes → Is $0 \leq a_n \leq b_n$
  - Yes → $\sum a_n$ Converges
- No → Is $0 \leq b_n \leq a_n$
  - Yes → $\sum a_n$ Diverges

Limit Comparison Test

Pick $\{b_n\}$. Does $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, c finite & $a_n, b_n > 0$?

- Yes → Does $\sum_{n=1}^{\infty} b_n$ converge?
  - Yes → $\sum a_n$ Converges
  - No → $\sum a_n$ Diverges

Integral Test

Does $a_n = f(n)$, $f(x)$ is continuous, positive & decreasing on $[a, \infty)$?

- Yes → Does $\int_a^{\infty} f(x)dx$ converge?
  - Yes → $\sum_{n=a}^{\infty} a_n$ converges
  - No → $\sum a_n$ Diverges

Ratio Test

Is $\lim_{n\to\infty} |a_{n+1}/a_n| \neq 1$?

- Yes → Is $\lim_{n\to\infty} |a_{n+1}/a_n| < 1$?
  - Yes → $\sum a_n$ Abs. Conv.
  - No → $\sum a_n$ Diverges

Root Test

Is $\lim_{n\to\infty} \sqrt[n]{|a_n|} \neq 1$?

- Yes → Is $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$?
  - Yes → $\sum a_n$ Abs. Conv.
  - No → $\sum a_n$ Diverges